Modification of the High Schmidt Number Solution for Laminar Boundary Layer Mass Transfer

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There have been several recent investigations (*1, 2, 3, 4*) concerned with the calculation of mass transfer in laminar boundary layer flows at high Schmidt numbers. It has been shown that even for finite interfacial velocities, a closed form solution of the diffusion equation valid for arbitrary geometries can be obtained. However, this solution is restricted to the case where the Schmidt number is large, which is generally true for diffusion in liquids. In particular, the minimum Schmidt number for which this solution can be applied increases as the surface mass transfer increases (as $|B|$ increases). Since it is known that the high Schmidt number results work well for Schmidt numbers of order unity when $B = 0$, it is natural to inquire into the possibility of modifying these results so that they could be applied at finite N_{Sc} even for fairly high mass transfer rates. Such a finite Schmidt number correction for the case of a zero pressure gradient is obtained below.

For constant properties, the laminar boundary layer momentum, continuity, and diffusion equations become

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial u^2}{\partial y^2} + U\frac{dU}{dx} \quad (1)$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (2)$$

$$u\frac{\partial W}{\partial x} + v\frac{\partial W}{\partial y} = D\frac{\partial^2 W}{\partial y^2} \quad (3)$$

For sufficiently high Schmidt numbers, it can be shown that the diffusion boundary layer becomes thin in relation to the momentum layer. Also, at higher Schmidt numbers the effect of the normal surface velocity on momentum transfer is diminished, and therefore at sufficiently large N_{Sc}, the diffusion equation may be solved with $u = (\tau_w(x)/\mu)y = \beta y$ as the velocity profile, where $\tau_w(x)$ is the zero surface velocity wall shear stress (*1, 2, 3, 4*). Since $v_w = -DG(\partial w/\partial y)_w$, substitution into Equation (3) gives

$$\beta y\frac{\partial W}{\partial x} - \left[DG\frac{\partial W}{\partial y}\bigg|_w + \frac{1}{2}\frac{d\beta}{dx}y^2\right]$$

$$\frac{\partial W}{\partial y} = D\frac{\partial^2 W}{\partial y^2} \quad (4)$$

When $(W - W_\infty)/(W_w - W_\infty) = \theta(\eta)$, is put where

$$\eta = \frac{\beta^{1/2}y}{(3D\int_0^x \beta^{1/2}\,dx)^{1/3}} \quad (5)$$

and $\theta'(0) \equiv b$, Equation (4) may be transformed into

$$\theta'' + \theta'[\eta^2 + Bb] = 0 \quad (6)$$

where $B \equiv G(W_w - W_\infty)$.

With the conditions $\theta(0) = 1$ and $\theta(\infty) = 0$ the solution of Equation (6) becomes

$$-\frac{1}{b} = \int_0^\infty e^{-\left(\frac{\lambda^3}{3} + Bb\lambda\right)}\,d\lambda \quad (7)$$

If the mass transfer coefficient is defined by

$$h_D = \frac{D}{W_\infty - W_w}\frac{\partial W}{\partial y}\bigg|_w \quad (8)$$

then Equations (5) and (7) yield

$$\left(\frac{h_D}{h_{D0}}\right)_\infty = \frac{-b\Gamma(1/3)}{3^{2/3}} = \frac{-b}{0.7763} \quad (9)$$

The quantity h_{D0} is the coefficient for the case of a negligibly small surface velocity. The subscript ∞ indicates that the result is applicable only at sufficiently large Schmidt numbers. The quantity h_{D0} is given by

$$\frac{h_{D0}}{D} = \frac{3^{1/3}}{\Gamma(1/3)}\frac{\beta^{1/2}}{(D\int_0^x \beta^{1/2}\,dx)^{1/3}} \quad (10)$$

These results have been derived by a number of authors (*1, 2, 3, 4*). The total surface mass flux of the diffusing component is given by $\dot{m}_1 = \rho h_D (W_w - W_\infty)(1 + GW_w)$. Values of $(h_D/h_D{}^0)_\infty$ are given as a function of B in reference 5. The relationship is shown graphically in Figure 1.

The above results apply for mass transfer from arbitrary two-dimensional bodies with finite surface velocities as long as the Schmidt number is sufficiently large. For $B = 0$, it is known (*6*) that Equation (10) gives fair results even for Schmidt numbers of order unity. To extend the usefulness of the high Schmidt number solution, the above results will be modified approximately so that they will be applicable at lower Schmidt numbers for a given B.

The situation considered is flow past a flat plate. Since the asymptotic results apply well at moderate Schmidt numbers if $B = 0$, it appears that they might be applicable for moderate Schmidt numbers when $B \neq 0$ provided that an improved $\beta(x)$ be used. Presumably an improved $\beta(x)$ should account to some extent for the effect of mass transfer on the wall shear stress.

The procedure is to find an appropriate expression for the wall shear stress valid at large Schmidt numbers. Then the modified results should reduce to the asymptotic results at sufficiently high Schmidt numbers. Hopefully the modified results will be reliable for high mass transfer rates at fairly low Schmidt numbers. At large Schmidt numbers, the asymptotic results apply reasonably well, and thus the quantity v_w may be written as

$$v_w = -DG\frac{\partial W}{\partial y}\bigg|_w = -\frac{DBb}{l} \quad (11)$$

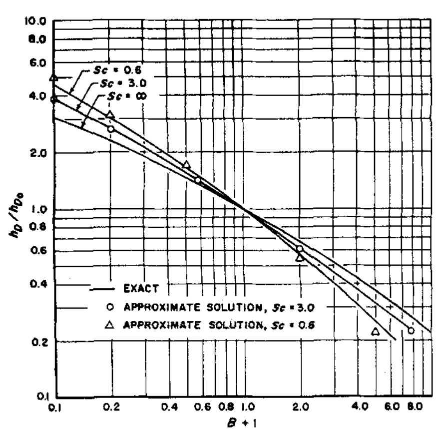

Fig. 1. Comparison of exact and modified results for mass transfer with U = constant.

Table 1. Surface Velocity Correction for Mass Transfer with $\dfrac{dp}{dx} = 0$

Outward mass transfer $[i'(0)/i'(0)_0]^{1/3} \leftarrow +$	$\dfrac{B(h_D/h_{D0})_\infty}{N_{Sc}^{2/3}}$	Inward mass transfer $- \rightarrow [i'(0)/i'(0)_0]^{1/3}$
0.981	0.1	1.019
0.962	0.2	1.038
0.904	0.5	1.092
0.806	1.0	1.181
0.613	2.0	1.343
0.436	3.0	1.486
0.183	5.0	1.726

where $l = \dfrac{\{3D \int_0^x \beta^{1/2}\, dx\}^{1/3}}{\beta^{1/2}}$ and b is given by Equation (7). We now take $u = \beta(x)y$ where β is $\dfrac{\tau w(x)}{\mu}$ for $B = 0$. Then by continuity, $v = v_w - \dfrac{d\beta}{dx}\dfrac{y^2}{2}$. When these relations are substituted into Equation (1) (for $\dfrac{dU}{dx} = 0$) one obtains

$$\beta y \frac{\partial u}{\partial x} - \left[\frac{DBb}{l} + \frac{d\beta}{dx}\frac{y^2}{2}\right]\frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} \quad (12)$$

If $\dfrac{u}{U} = i(\eta)$ and $\eta = \dfrac{y}{l}$ are used, Equation (12) becomes

$$- i'(\eta)\,[\eta^2 + Bb] = i''(\eta)\, N_{Sc} \quad (13)$$

The quantity of interest here is the corrected shear stress, $\beta_{corr.}$ With the boundary conditions $i(\infty) = 1$ and $i(0) = 0$, the following expression for $\beta_{corr.}$ is obtained:

$$\frac{\beta_{corr.}}{\beta} = \frac{i'(0)}{i'(0)_0} = \frac{\Gamma(1/3)/3^{2/3}}{\int_0^\infty e^{-\left[\frac{\lambda^3}{3} + \frac{Bb\lambda}{Sc^{2/3}}\right]}\, d\lambda} \quad (14)$$

According to Equation (9) this may also be written as

$$\frac{\beta_{corr.}}{\beta} = \frac{1.288}{\int_0^\infty e^{-\left[\frac{\lambda^3}{3} - \frac{0.7736B(h_D/h_{D_0})_\infty \lambda}{N_{Sc}^{2/3}}\right]}\, d\lambda} \quad (15)$$

Thus it is seen that the correction to the high Schmidt number solution depends only on the quantity $\dfrac{B(h_D/h_{D0})_\infty}{N_{Sc}^{2/3}}$.

Moreover the integral in Equation (15) is of the same form as that of Equation (7), and therefore numerical values of the latter integral can be obtained directly from a knowledge of values of the former. Table 1 contains numerical values which may be used in practical applications. If $\beta_{corr.}$ is now put into Equations (4) and (5) in place of β

$$\frac{h_D}{h_{D0}} = \left(\frac{h_D}{h_{D0}}\right)_\infty \left\{\frac{i'(0)}{i'(0)_0}\right\}^{1/3} \quad (16)$$

The correction goes to unity when either $B \rightarrow 0$ or $N_{Sc} \rightarrow \infty$, as expected. For $v_w = 0$, Lighthill (6) has shown that a modification of the above procedure may be used to solve the momentum equation even for arbitrary geometries.

The comparison of the modified high Schmidt number results with exact solutions will now be considered. The nature of the correction seems to indicate that it will give best results for either sufficiently low values of B or high values of N_{Sc}. In Figure 1, the modified results are compared to exact results for the flat plate at $N_{Sc} = 3.0$ and $N_{Sc} = 0.6$ for various values of B. The agreement is virtually exact for $N_{Sc} = 3.0$ and is surprisingly good for $N_{Sc} = 0.6$. At $N_{Sc} = 2.0$ and $-0.9 < B < 8$ the modified and exact results are nearly identical. The agreement for $N_{Sc} = 1.0$ is indicated in Figure 2. Presumably the agreement would also be good for higher Schmidt numbers.

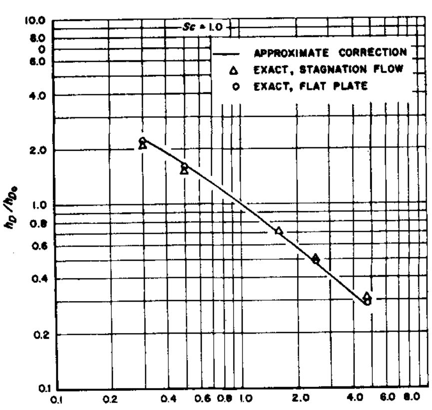

Fig. 2. Comparison of exact and modified results for mass transfer with U = Cx.

The good agreement between the modified high Schmidt number results and exact calculations for $N_{Sc} \geqq 0.6$ at high mass transfer rates is evidently owing to the insensitivity of $\dfrac{h_D}{h_{D0}}$ to Schmidt number variations. Since the high Schmidt number results are valid for arbitrary geometries and the flat plate corrected results join the asymptotic results smoothly at high N_{Sc}, it would seem that the corrected results might apply reasonably well for arbitrary geometries at sufficiently high Schmidt numbers. In Figure 2 values of (h_D/h_{D0}) for stagnation flow at $N_{Sc} = 1.0$ are compared to those predicted by the approximate correction. The agreement is fairly good, although less satisfactory than for the flat plate.

The finite Schmidt number correction outlined above may also be applied to the problem of constant property simultaneous heat and mass transfer at large Schmidt and Prandtl numbers. For this case, if $h(t_w - t_\infty) = -k \dfrac{\partial t}{\partial y}\Big|_w$ one obtains

$$\left(\frac{h}{h_0}\right)_\infty = \frac{1.288}{\int_0^\infty e^{-\left[\frac{\lambda^3}{3} - 0.7763\left(\frac{h_D}{h_{D_0}}\right)_\infty B N_{Le}^{2/3}\lambda\right]} d\lambda} \quad (17)$$

The quantity $\left(\dfrac{h}{h_0}\right)_\infty$ is the ratio of coefficients with and without a surface velocity at large N_{Sc} and N_{Pr}. The quantity h_0 is given by

$$\frac{h_0}{k} = \frac{\beta^{1/2}\,(3^{1/3})}{\Gamma(1/3)\,\{\alpha \int_0^x \beta^{1/2}\, dx\}^{1/3}} \quad (18)$$

It is clear that the same finite Schmidt number correction derived for pure mass transfer also applies here. Then

$$\frac{h}{h_0} = \left(\frac{h}{h_0}\right)_\infty \left\{\frac{i'(0)}{i'(0)_0}\right\}^{1/3} \quad (19)$$

For $N_{Le} \equiv 1$, $h_D/h_{D0} = h/h_0$. For $N_{Le} \neq 1$, h depends in N_{Pr}, N_{Sc} and N_{Le}; however, since $N_{Le} \equiv N_{Pr}/N_{Sc}$, h depends only upon two of these three parameters.

NOTATION

a = $\dot{m}_2/\dot{m}_1$
B = $G(W_w - W_\infty)$
b = defined by Equation (6)
C_p = heat capacity
D = diffusion coefficient
G = $(1 + a)/[1 - W_w(1 + a)]$
h = heat transfer coefficient, $h = \dfrac{k}{t_\infty - t_w}\dfrac{\partial t}{\partial y}\Big|_w$

$(h/h_0)_\infty$ = ratio of heat transfer coefficients for large Sc and Pr
h_D = mass transfer coefficient, $h_D = \dfrac{D}{W_\infty - W_w} \dfrac{\partial W}{\partial y}\Big|_w$
$(h_D/h_{D0})_\infty$ = ratio of mass transfer coefficients for large Sc
k = thermal conductivity
$l(x)$ = $\{3D \int_0^x \beta^{1/2}\, dx\}^{1/3}/\beta^{1/2}$
$\dot{m}_{1\,\text{diffusion}}$ = surface mass flux of component 1 by diffusion
$\dot{m}_1$ = total surface mass flux of component 1
t = temperature
t_∞ = free stream temperature
u = longitudinal velocity component
$U(x)$ = local free stream velocity
v = transverse velocity component
W = mass fraction of component 1
W_∞ = free stream mass fraction of component 1
x = longitudinal coordinate
y = transverse coordinate

Greek Letters

α = thermal diffusivity, $k/C_p\rho$
β = τ_w/μ for $B = 0$
$\beta_{corr.}$ = τ_w/μ corrected for effects of surface velocity
η = $y/l(x)$
μ = dynamic viscosity
ν = kinematic viscosity, μ/ρ
ρ = density
τ_w = surface shear stress
N_{Le} = Lewis number, N_{Pr}/N_{Sc}
N_{Pr} = Prandtl number, $\mu C_p/k$
N_{Sc} = Schmidt number, $\mu/\rho D$

Subscripts

0 = condition of $B = 0$
w = conditions at the wall

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Equilibrium Flow of a General Fluid Through a Cylindrical Tube

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In a recent communication to the editor (8) questions were raised about the results of work presented in a previously published paper (6). These questions will be answered in the following discussion.

The majority of the questions raised in reference 8 may be dismissed if one accepts the proposition that equilibrium or fully developed flow did exist in the tests reported in reference 6 for a tube with an L/D ratio of about 48, within the range of velocity gradients reported in Figure 4 of the same reference.

In opposition to references 2 and 4, which suggest that the L/D ratio necessary to attain equilibrium flow for viscoelastic fluids must be well over 100, the results of work of several other investigators (1, 3, 5, 7) indicate that equilibrium flow is attained at L/D tube ratios of much less than 48. Furthermore, the L/D ratio required to attain equilibrium flow should be a property of the test material as well as of the existing flow conditions.

The fact that some experimenters have found that σ_1 is smaller than τ does not necessarily contradict the results shown in Figure 4 (6), since at $-\left[\dfrac{dv_z}{dr}\right] < \sim 45$ sec.$^{-1}$, $\sigma_1 < -\tau$. In accordance with this the material function σ_1 should increase more rapidly with increasing velocity gradient than should the material function τ. Thus above some limiting velocity gradient, which will also depend on the specific test material, σ_1 would be expected to be greater in magnitude than τ.

With regard to the boundary condition at the tube exit, $[p_{ro}]_L$, since at $z = L$, and $r = 0$, $\left[\dfrac{dv_z}{dr}\right] = 0$, it follows that $\dfrac{d[p_{ro}]_L}{da} = 0$, although $[p_{ro}]_L$ may not be zero. Equation (18) of reference 6 should be written as

$$[p_{ro}]_L = -[p'']_{r=0,L} = [p_{zL}]_{r=0} \equiv h \qquad (18a)$$

where h is a parameter, which does not depend on the velocity gradient $\left[\dfrac{dv_z}{dr}\right]$.

In place of Equations (19) and (20) in reference 6

$$P_{RL} = -[p_{rR}]_L = \int_0^R \xi^{-1}\sigma_1 \left[\tau^{-1}\left(\frac{1}{2}a\xi\right)\right] d\xi - h \qquad (19a)$$

and

$$P_{RL} = \int_0^{\frac{1}{2}aR} \xi^{-1}\sigma_1[\tau^{-1}(\xi)]d\xi - h \qquad (20a)$$

The significance* of the parameter h can be seen by an evaluation of Equation (22) in reference 6 at $r = R$ and $z = L$ and a comparison with Equation (26) in reference 6 which reads

$$\sigma_2\left[\tau^{-1}\left(\frac{1}{2}aR\right)\right] = \sigma_1\left[\tau^{-1}\left(\frac{1}{2}aR\right)\right] + P_{RL} + \frac{4}{aR^2}\frac{d}{da}\left[\int_0^{\frac{1}{2}aR} p_{zL}\xi d\xi\right] \qquad (26)$$

when one finds that

$$\frac{4}{aR^2}\frac{d}{da}\left[\int_0^{\frac{1}{2}aR} p_{zL}\xi d\xi\right] = [p_{zL}]_R$$

It can also be shown that when the

* The introduction of the parameter h does not affect any other equations of reference 6, with the exception of Equation (33), nor does it affect the experimental measurements, computations, and conclusions drawn from these measurements. Equation (33) is, of course, valid for $h = 0$.